1. Harvard Law School courses often have assigned seating to facilitate the “Socratic method.” Suppose that there are 100 first year Harvard Law students, and each takes two courses: Torts and Contracts. Both are held in the same lecture hall (which has 100 seats), and the seating is uniformly random and independent for the two courses.

(a) Find the probability that no one has the same seat for both courses (exactly; you should leave your answer as a sum).   
(b) Find a simple but accurate approximation to the probability that no one has the same seat for both courses.   
(c) Find a simple but accurate approximation to the probability that at least two students have the same seat for both courses.

1. There are 100 passengers lined up to board an airplane with 100 seats (with each seat assigned to one of the passengers). The first passenger in line crazily decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes his or her assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in his or her assigned seat?

Answer:

1. (a) Let's consider the first student. The probability that they don't have the same seat for both courses is 1, since they haven't taken any courses yet. Now consider the second student. The probability that they also don't have the same seat for both courses is 99/100, since there are 99 seats left and only one of them was taken by the first student. For the third student, the probability is 98/100, and so on. Therefore, the probability that no one has the same seat for both courses is:

1 × 99/100 × 98/100 × ... × 2/100 × 1/100 = Π\_{i=1}^{100} (101-i)/100! ≈ 0.00000229.

(b) A simple approximation to this probability can be obtained using the inclusion-exclusion principle. Let A\_i be the event that the i-th student has the same seat for both courses. Then the probability that at least one student has the same seat for both courses is:

P(A\_1 ∪ A\_2 ∪ ... ∪ A\_100) ≤ Σ\_{i=1}^{100} P(A\_i) = 100/100^2 = 1/100.

To get an approximation for the probability that no one has the same seat for both courses, we can use the complement rule and subtract this from 1:

P(no one has the same seat) ≈ 1 - 1/100 = 0.99.

(c) A simple approximation for the probability that at least two students have the same seat for both courses can be obtained using the birthday problem. Let's consider the 100 students as 100 "days" and the seat assignments as "people" being born on those days. The probability that at least two students have the same seat for both courses is the same as the probability that at least two people have the same birthday.

The probability of no two people having the same birthday is:

365/365 × 364/365 × 363/365 × ... × 266/365 ≈ 0.0000000000875.

Therefore, the probability of at least two people having the same birthday (or seat assignment) is approximately 1 - 0.0000000000875 ≈ 1.

2.

We can approach this problem using conditional probability. Let A\_i be the event that the i-th passenger gets to sit in their assigned seat, and let B\_i be the event that the i-th passenger's assigned seat is still available when they get to their seat. Then we want to find P(A\_100), the probability that the last passenger gets to sit in their assigned seat.

Clearly, P(A\_1) = 1 since the first passenger can sit in any seat. For 2 ≤ i ≤ 99, we have:

P(A\_i) = P(A\_i | B\_i) P(B\_i) + P(A\_i | not B\_i) P(not B\_i).

If the i-th passenger's assigned seat is available, then they will sit in it, so P(A\_i | B\_i) = 1. The probability that the i-th passenger's assigned seat is available is P(B\_i) = 1/i, since the first i passengers can sit in any seat, and the i-th seat is still available with probability 1/i.

If the i-th passenger's assigned seat is not available, then they will randomly choose one of the remaining available seats. There are i-1 seats still available at this point, so the probability that they choose the i-th seat is 1/(i-1). Therefore, P(A\_i | not B\_i) = 1/(i-1).

Finally, we have:

P(A\_100) = P(A\_100 | B\_100) P(B\_100) + P(A\_100 | not B\_100) P(not B\_100)

= 1 × 1/100 + 0 × 99/100

= 1/100.

Therefore, the probability that the last passenger gets to sit in their assigned seat is 1/100.